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Groups of automorphisms of the canonical commutation and anticommutation relations

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Abstract. Observables of supersymmetric quantum mechanics are coded by taking the antisymmetric tensor product with anticommuting parameters. Next we define superunitary transformations, which mix bosonic and fermionic degrees of freedom, in order to construct automorphisms of the canonical (anti)commutation relations (C(A)CR). Conversely, every automorphism of the C(A)CR is implemented by an essentially unique superunitary transformation.

1. Introduction

The concept of supersymmetry was invented and first developed within the framework of relativistic quantum field theory [1]. Nevertheless, applications of supersymmetric algebras to systems with a finite number of degrees of freedom lead to many physically relevant problems [2]. For extensive surveys of physical treatments of supersymmetry see, for example, the reviews [3].

An axiomatic formulation of supersymmetric quantum mechanics (SSQM) has been introduced in [4], which can be rewritten conveniently in terms of sesquilinear forms [5]. Here we are working in the Hilbert space of f bosonic and f fermionic degrees of freedom, which is isomorphic to the tensor product of the Hilbert space of wavefunctions $L^2(\mathbb{R}^f)$ times the Grassmann algebra of f anticommuting variables.

In order to investigate transformations, which mix bosonic and fermionic degrees of freedom, an additional Grassmann algebra of g anticommuting parameters is introduced. Next we take the antisymmetric tensor product of that algebra with the C^* algebra of bounded operators on our Hilbert space. The notion of a scalar product is extended to matrix elements of such coded operators with the help of the van Hove rule [6]. Unbounded operators are restricted to invariant domains and, similarly, coded by these anticommuting parameters.

Next we define groups of superunitary transformations and use them in order to construct automorphisms of the canonical (anti)commutation relations (C(A)CR). Conversely, we show that any two representations of the C(A)CR by coded operators, defined on suitable domains, are connected by an essentially unique superunitary transformation. In that way we extend von Neumann's uniqueness theorem [7] to the linear

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combination of bosonic and fermionic observables. These superunitary transformations are then combined with unitary transformations of the fermionic oscillator representation of the C(A)CR.

We note that various aspects of supergroups have been dealt with in recent publications [8].

2. Fermionic oscillator representation of the C(A)CR

We use the Grassmann algebra $\mathscr{G}_f$ of polynomials in the pairwise anticommuting variables C_k, $k=1,\ldots,f$, over the field of complex numbers $\mathbb{C}$, in order to describe systems with f fermionic degrees of freedom. Any element $C \in \mathscr{G}_f$ can be expanded into 2^f monomials [9]

$$C = \gamma_0 I_f + \sum_{1\leqslant i_1<\ldots<i_p\leqslant f} \gamma_{i_1\ldots i_p} C_{i_1}\ldots C_{i_p} \qquad \gamma_0, \gamma_{i_1\ldots i_p} \in \mathbb{C} \tag{2.1}$$

where I_f denotes the unit element of the associative algebra $\mathscr{G}_f$. Through the definition of a degree, i.e. deg $C=0$ (deg $C=1$), if C is a linear combination of monomials with even (odd) p, $\mathscr{G}_f$ becomes an associative superalgebra. $\gamma_0 I_f$ gets degree zero, too. Elements of $\mathscr{G}_f$, which are either even or odd, are called homogeneous.

The derivative from the left with respect to C_k is defined via linear extension of

$$D_k C_{i_1}\ldots C_{i_p} := \sum_{n=1}^{p} (-)^{n-1}\delta_{ki_n} C_{i_1}\ldots \cancel{C}_{i_n}\ldots C_{i_p} \qquad D_k I_f = 0 \qquad k=1,\ldots,f \tag{2.2}$$

where the notation $\cancel{C}_{i_n}$ means, that C_{i_n} has to be omitted. For a homogeneous element of degree deg C, we get

$$D_k(CC') = (D_kC)C' + (-1)^{\deg C} C(D_kC') \qquad k=1,\ldots,f \tag{2.3}$$

which shows that the endomorphism D_k is a graded derivative of $\mathscr{G}_f$.

The Grassmann algebra $\mathscr{G}'_f$ of polynomials in D_k, $k=1,\ldots,f$, is combined with the algebra $\mathscr{G}_f$, fulfilling the canonical anticommutation relations (CAR)

$$[C_i, C_k]_+ = 0 \qquad [D_i, D_k]_+ = 0 \qquad [C_i, D_k]_+ = \delta_{ik} I_f \qquad i,k=1,\ldots f \tag{2.4}$$

and yields the Clifford algebra $\mathscr{C}_f$ of polynomials in the $2f$ variables $C_k + D_k$ and $\mathrm{i}(C_k - D_k)$, $k=1,\ldots,f$, over $\mathbb{C}$.

With the scalar product [5]

$$\langle C|C'\rangle := C_0^* C_0' + \sum_{1\leqslant i_1<\ldots<i_p\leqslant f} C^*_{i_1\ldots i_p} C'_{i_1\ldots i_p} \tag{2.5}$$

$\mathscr{G}_f$ becomes isomorphic to the f-fold tensor product $\bigotimes^f \mathbb{C}^2$. Its basic elements can be represented by the Klein-Jordan-Wigner transformed Pauli matrices

$$\begin{aligned} C_k &= \bigotimes^{k-1} (-\sigma^3)\otimes\sigma^+ \bigotimes^{f-k} \varepsilon_2 \qquad k=2,\ldots,f \\ C_1 &= \sigma^+ \bigotimes^{f-1} \varepsilon_2 \\ \varepsilon_2 &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.6}$$

This scalar product implies that $D_k = C_k^+$, $k=1,\ldots,f$.

In order to combine these f fermionic degrees of freedom with an equal number of bosonic degrees of freedom in the spirit of SSQM, we define the tensor product $\mathscr{H}_f = \mathscr{H}_0 \otimes \mathscr{G}_f$ of $\mathscr{G}_f$ with a separable Hilbert space $\mathscr{H}_0$. $\mathscr{H}_f$ can be decomposed according to the Z_2 grading of $\mathscr{G}_f$. Let us denote the projection operators onto bosonic and fermionic states by N_0 and N_1, respectively. The Klein operator is then defined by $K = N_0 - N_1$ [6]. The scalar product of states $\Psi, \Phi \in \mathscr{H}_f$ is given by

$$\begin{aligned} \langle\Phi|\Psi\rangle &= \langle\phi_0|\psi_0\rangle + \sum_{i \leqslant i_1 < \ldots < i_p \leqslant f} \langle\phi_{i_1\ldots i_p}|\psi_{i_1\ldots i_p}\rangle \\ \Psi &= \psi_0 I_f + \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant f} \psi_{i_1\ldots i_p} C_{i_1} \ldots C_{i_p} \qquad \psi_0, \psi_{i_1\ldots i_p} \in \mathscr{H}_0 \end{aligned} \tag{2.7}$$

where Φ has a similar decomposition as Ψ.

The separable Hilbert space $\mathscr{L}_f = L^2(\mathbb{R}^f) \otimes \mathscr{G}_f$ becomes isomorphic to the $2f$-fold tensor product $\bigotimes^f (L^2(\mathbb{R}^1) \otimes \mathbb{C}^2)$, if one takes for $\mathscr{H}_0 = L^2(\mathbb{R}^f)$ the Hilbert space for f bosonic degrees of freedom. The closed operators in $L^2(\mathbb{R}^f)$

$$\begin{aligned} &\sqrt{2}\, B_k := X_k + iP_k \qquad \sqrt{2}\, B_k^\dagger := X_k - iP_k \\ &\operatorname{dom} B_k = \operatorname{dom} B_k^\dagger = \operatorname{dom} X_k \cap \operatorname{dom} P_k \qquad k = 1, \ldots, f \end{aligned} \tag{2.8}$$

fulfil the canonical commutation relations (CCR)

$$\begin{aligned} &[B_k, B_k^\dagger]_- \subset I_x \qquad \operatorname{dom} B_k B_k^\dagger = \operatorname{dom} B_k^\dagger B_k \\ &[B_i, B_k]_- = 0 \qquad [B_i, B_k^\dagger]_- = 0 \qquad i \neq k = 1, \ldots, f \end{aligned} \tag{2.9}$$

in the sense of spectral families of position operators X_k and momentum operators P_k. In equations (2.9) I_x denotes the identity mapping of $L^2(\mathbb{R}^f)$.

In the following, we shall use the shorthand writing B_k and C_k for $B_k \otimes I_f$ and $I_x \otimes C_k$, and similar abbreviations will be used for closed operators in $L^2(\mathbb{R}^f)$ and endomorphisms of $\mathscr{G}_f$. The CCR (2.9) are combined with the CAR (2.4) in $\mathscr{L}_f$ to the canonical (anti-) commutation relations (C(A)CR)

$$\begin{aligned} &[B_i, B_k^\dagger]_- \subset \delta_{ik} I_x \qquad [C_i, C_k^\dagger]_+ = \delta_{ik} I \qquad i, k = 1, \ldots, f \\ &[B_i, B_k]_- = 0 \qquad [C_i, C_k]_+ = 0 \qquad [B_i, C_k]_- = 0 \qquad [B_i, C_k^\dagger]_- = 0 \end{aligned} \tag{2.10}$$

which hold in the sense of spectral resolutions. Von Neumann's theorem asserts that every irreducible representation of the C(A)CR in a separable Hilbert space is unitarily equivalent to the 'fermionic oscillator representation' (2.10) [7]. For a more detailed statement see [5], for example.

The fermionic oscillator model is defined by the Hamilton operator

$$H := \sum_{k=1}^{f} (B_k^\dagger B_k + C_k^\dagger C_k) = \tfrac{1}{2} \sum_{k=1}^{f} (P_k^2 \dotplus X_k^2 - \sigma_k^3) \geqslant 0 \tag{2.11}$$

where the Klein–Jordan–Wigner transformation was used in (2.11). The $\dotplus$ on the RHS of (2.11) denotes the form sum [10]. The supersymmetric structure of this model is well known.

3. Anticommuting parameters as coefficients of SSQM

An axiomatic formulation of SSQM can be conveniently given in terms of sesquilinear forms [5]. One thereby defines self-adjoint supercharges Q_n, $n = 1, \ldots, N$, which act in the separable Hilbert space $\mathscr{H}$. Let N_0 and N_1 be the projection operators onto even and odd states with $N_0+N_1=1$, and define the Klein operator as $K=N_0-N_1$. Then one requires, that

$$\begin{aligned} &Q_n^2=H \qquad \operatorname{dom} Q_n=\operatorname{dom} H^{1/2} \qquad n=1,\ldots,N \\ &\langle Q_n\Phi|Q_m\Psi\rangle+\langle Q_m\Phi|Q_n\Psi\rangle=0 \qquad n\neq m=1,\ldots,N \\ &\langle Q_n\Phi|K\Psi\rangle+\langle K\Phi|Q_n\Psi\rangle=0 \qquad \Psi,\Phi\in\operatorname{dom} H^{1/2}. \end{aligned} \tag{3.1}$$

The supercharges map states from $N_0\mathscr{H}$ to $N_1\mathscr{H}$ and vice versa. H will be reduced by the bosonic subspace $N_0\mathscr{H}$ as well as by the fermionic subspace $N_1\mathscr{H}$.

In order to construct automorphisms of the C(A)CR, which combine linearly bosonic and fermionic operators, and in order to respect the Z_2 grading, one introduces anticommuting parameters: Let $\mathscr{D}_g$ be the Grassmann algebra of polynomials in the variables Θ_i, $i=1,\ldots,g$, with complex coefficients. We construct the skew-symmetric tensor product $\mathscr{D}_g\otimes B(\mathscr{H})$ of $\mathscr{D}_g$ with the C^* algebra of bounded operators on $\mathscr{H}$, which implies a Z_2 gradation. The orthogonal decomposition of $\mathscr{H}$ into bosonic and fermionic states enables one to consider $B(\mathscr{H})$ as an associative superalgebra [11], since any $A\in B(\mathscr{H})$ can be decomposed as $A=(N_0AN_0+N_1AN_1)+(N_0AN_1+N_1AN_0)$. The composition law

$$(\Theta\otimes A)(\Theta'\otimes A'):=(-1)^{\deg A.\deg\Theta'}(\Theta\Theta')\otimes(AA') \qquad A,A'\in B(\mathscr{H}),\Theta,\Theta'\in\mathscr{D}_g \tag{3.2}$$

for homogeneous elements A and Θ' can be linearly extended to $\mathscr{D}_g\otimes B(\mathscr{H})$; it can be represented by an endomorphism of the algebraic tensor product $\mathscr{D}_g\otimes\mathscr{H}$ according to

$$(\Theta\otimes A)(\Theta'\otimes\Psi):=(-1)^{\deg A.\deg\Theta'}(\Theta\Theta')\otimes(A\Psi) \qquad \Psi\in\mathscr{H}. \tag{3.3}$$

The elements of such tensor products can be expanded into 2^g basis elements of $\mathscr{D}_g$, according to

$$\begin{aligned} &I_g\otimes A_0+\sum_{1\leq i_1<\ldots<i_p\leq g}\Theta_{i_1}\ldots\Theta_{i_p}\otimes A_{i_1\ldots i_p}\in\mathscr{D}_g\otimes B(\mathscr{H}) \qquad A_{i_1\ldots i_p}\in B(\mathscr{H}) \\ &I_g\otimes\Psi_0+\sum_{1\leq i_1<\ldots<i_p\leq g}\Theta_{i_1}\ldots\Theta_{i_p}\otimes\Psi_{i_1\ldots i_p}\in\mathscr{D}_g\otimes\mathscr{H} \qquad \Psi_{i_1\ldots i_p}\in\mathscr{H} \end{aligned} \tag{3.4}$$

where I_g denotes the unit element of $\mathscr{D}_g$.

In order to extend adjointness to the tensor product (3.2), one defines an involution on $\mathscr{D}_g$ by

$$\left(c_0I_g+\sum_{1\leq i_1<\ldots<i_p\leq g}c_{i_1\ldots i_p}\Theta_{i_1}\ldots\Theta_{i_p}\right)^*:=c_0^*I_g+\sum_{1\leq i_1<\ldots<i_p\leq g}c^*_{i_1\ldots i_p}\Theta_{i_p}\ldots\Theta_{i_1} \tag{3.5}$$

especially $\Theta_i^*=\Theta_i$ and $(\Theta_i\Theta_k)^*=\Theta_k\Theta_i$, i, $k=1,\ldots,g$. This involution is extended to $\mathscr{D}_g\otimes B(\mathscr{H})$ by

$$(\Theta\otimes A)^*:=(I_g\otimes A^\dagger)(\Theta^*\otimes I) \qquad \Theta\in\mathscr{D}_g;A\in B(\mathscr{H}) \tag{3.6}$$

where I denotes the identity mapping of $\mathscr{H}$. In the following we shall suppress the symbol $\otimes$, and write

$$\Theta\otimes A=(\Theta\otimes I)(I_g\otimes A)\equiv\Theta A \qquad (I_g\otimes A)(\Theta\otimes I)\equiv A\Theta \tag{3.7}$$

and $\Theta A=(-1)^{\deg A.\deg\Theta}A\Theta$ holds for homogeneous elements A and Θ. With that convention, (3.6) is rewritten as $(\Theta A)^*=A^\dagger\Theta^*$.

For the special case $\mathscr{H}=\mathscr{H}_f$, the linear combinations of $I_g\otimes I_f$, $I_g\otimes C_{k_1}\dots C_{k_p}$, $\Theta_{i_1}\dots\Theta_{i_q}\otimes I_f$ and $\Theta_{i_1}\dots\Theta_{i_q}\otimes C_{k_1}\dots C_{k_p}$, $1\leq i_1<\dots<i_q\leq g$, $1\leq k_1<\dots<k_p\leq f$, with complex coefficients yield the Grassmann algebra with basis $\{\Theta_1\dots\Theta_g, C_1\dots C_f\}$.

Definition (3.3) is naturally generalised to unbounded operators by defining

$$\begin{aligned}\operatorname{dom}\Big(I_gA_0+\sum_{\{i\}}\Theta_{i_1}\dots\Theta_{i_p}A_{\{i\}}\Big)\\ :=\Big\{I_g\Psi_0+\sum_{\{j\}}\Theta_{j_1}\dots\Theta_{j_q}\Psi_{\{j\}}|\Psi_0\in\bigcap_{\{i\}}\operatorname{dom}A_{\{i\}},\\ \Psi_{\{j\}}\in\operatorname{dom}A_{\{i\}}\text{ for }\{i\}\bigcap\{j\}=\varnothing\Big\}.\end{aligned}\tag{3.8}$$

Such tensor products shall be called coded operators acting on coded states. In the following, the domains of densely defined operators A and their adjoints are assumed to be graded, which means that $\Psi\in\operatorname{dom}A$ implies $K\Psi\in\operatorname{dom}A$ and also $K\operatorname{dom}A^{\dagger}\subseteq\operatorname{dom}A^{\dagger}$. As a special case we note that the domain of an even or odd operator is graded. Using the relation $\Theta A=KAK\Theta$, which holds for odd $\Theta\in\mathscr{D}_g$, one extends the operation of taking the adjoint to coded operators by the rules

$$\begin{aligned}\Big(I_gA_0+\sum_{\{i\}}\Theta_{i_1}\dots\Theta_{i_p}A_{\{i\}}\Big)^{\dagger}&:=A_0^{\dagger}I_g+\sum_{\{i\}}A_{\{i\}}^{\dagger}\Theta_{i_p}\dots\Theta_{i_1}\\ &=I_gA_0^{\dagger}+\sum_{p\text{ even}}\Theta_{i_p}\dots\Theta_{i_1}A_{\{i\}}^{\dagger}+\sum_{p\text{ odd}}\Theta_{i_p}\dots\Theta_{i_1}KA_{\{i\}}^{\dagger}K\end{aligned}\tag{3.9}$$

$$\Big(I_gA_0+\sum_{\{i\}}\Theta_{i_1}\dots\Theta_{i_p}A_{\{i\}}\Big)^{\dagger\dagger}=I_g\bar{A}_0+\sum_{\{i\}}\Theta_{i_1}\dots\Theta_{i_p}\bar{A}_{\{i\}}$$

for closable operators A_0, $A_{\{i\}}$ in $\mathscr{H}$.

The algebraic tensor product $\mathscr{D}_g\otimes\mathscr{H}$ is not equipped with a topology, because the anticommuting parameters Θ_i, $i=1,\dots,g$, are not considered as operators, but merely as coefficients. The scalar product on $\mathscr{H}\times\mathscr{H}$ can therefore be extended to the following mapping from $(\mathscr{D}_g\otimes\mathscr{H})^2$ into $\mathscr{D}_g$ [6]:

$$\begin{aligned}\Big\langle\sum_{\{i\}}\Theta_{i_1}\dots\Theta_{i_p}\Phi_{\{i\}}\Big|\sum_{\{j\}}\Theta_{j_1}\dots\Theta_{j_q}\Psi_{\{j\}}\Big\rangle\\ :=\sum_{\{i\}\cap\{j\}=\varnothing}\langle\Phi_{\{i\}}|\Theta_{i_p}\dots\Theta_{i_1}\Theta_{j_1}\dots\Theta_{j_q}\Psi_{\{j\}}\rangle\\ :=\sum_{\{i\}\cap\{j\}=\varnothing}\Theta_{i_p}\dots\Theta_{i_1}\Theta_{j_1}\dots\Theta_{j_q}\langle\Phi_{\{i\}}|K^{p+q}\Psi_{\{j\}}\rangle\end{aligned}\tag{3.10}$$

where we used the fact that $K^2=I$. The rules (3.9) can therefore be understood as adjointness relations of the form

$$\begin{aligned}\langle\Theta'\Psi'|\Theta A\Theta''\Psi''\rangle&=\langle A^{\dagger}\Theta^*\Theta'\Psi'|\Theta''\Psi''\rangle\\ &=\langle\Theta^*\Theta'K^{d+d'}A^{\dagger}K^{d+d'}\Psi'|\Theta''\Psi''\rangle\\ &=\Theta'^*\Theta\Theta''\langle K^{d''}A^{\dagger}K^{d+d'}\Psi'|\Psi''\rangle\\ &=\Theta'^*\Theta\Theta''\langle K^{d+d'}\Psi'|AK^{d''}\Psi''\rangle\end{aligned}\tag{3.11}$$

where the degrees of Θ, Θ', Θ'' are denoted by d, d', d''.

4. Groups of supertransformations

The Z_2 graded tensor product (3.2) can be used to construct groups of supertransformations. Let $A_{\mathscr{C}}$ be the associative superalgebra of linear operators G with a common invariant domain $\mathscr{C} = \text{dom}\, G$, which is dense in the separable Hilbert space $\mathscr{H}$; therefore $G\mathscr{C} \subseteq \mathscr{C}$. $\mathscr{H}$ is assumed to be Z_2 graded with Klein operator K, and $\mathscr{C}$ is assumed to be graded. By linear extension of the supercommutator [11]

$$[G, G'] := GG' - (-1)^{\deg G . \deg G'} G'G \qquad G, G' \in A_{\mathscr{C}} \tag{4.1}$$

where G and G' are homogeneous, $A_{\mathscr{C}}$ becomes the Lie superalgebra $L_{\mathscr{C}}$. The skew-symmetric tensor product $\mathscr{D}_g \otimes A_{\mathscr{C}}$ becomes an associative superalgebra, if we restrict states to be in $\mathscr{C}$ and linear operators to be in $A_{\mathscr{C}}$ in the definitions (3.2) and (3.3).

Let $A^*_{\mathscr{C}}$ be the subalgebra of linear operators $G \in A_{\mathscr{C}}$ such that dom $G^\dagger \supseteq \mathscr{C}$ and $G^\dagger \mathscr{C} \subseteq \mathscr{C}$. The corresponding Lie superalgebra will be denoted by $L^*_{\mathscr{C}}$. For the skew-symmetric tensor product $\mathscr{D}_g \otimes A^*_{\mathscr{C}}$, which is an associative superalgebra, one can define an involution by linear extension of

$$(\Theta G)^* := (\Theta G)^\dagger|_{\mathscr{D}_g \otimes \mathscr{C}} \qquad (\Theta G)^{**} = \Theta G \qquad G \in A^*_{\mathscr{C}}, \Theta \in \mathscr{D}_g. \tag{4.2}$$

The coded operator $T \in \mathscr{D}_g \otimes A^*_{\mathscr{C}}$ is called superunitary on $\mathscr{C}$, iff it fulfills the conditions

$$T = I_g \otimes I|_{\mathscr{C}} + \sum_{\{i\}} \Theta_{i_1} \dots \Theta_{i_p} T_{\{i\}} \qquad T^*T = TT^* = I_g \otimes I|_{\mathscr{C}}. \tag{4.3}$$

In (4.3) $T_{\{i\}}$ is assumed to be even (odd) for even (odd) p, such that T is an even linear bijection of the algebraic tensor product $\mathscr{D}_g \otimes \mathscr{C}$.

If T_1 and T_2 are superunitary, $T_1 T_2$ is superunitary too. The set of superunitary operators on $\mathscr{C}$ therefore becomes a group with unit element $I_g \otimes I|_{\mathscr{C}}$, and the inverse elements are obtained with the help of the involution defined above. For some fixed invariant common domain $\mathscr{C}$ this group will be denoted by $U_{\mathscr{C}}$.

Let Θ^k, $k = 1, \dots, q$, be homogeneous elements of $\mathscr{D}_g$ such that $(\Theta^k)^2 = 0$, and let G_k be out of $L_{\mathscr{C}}$ such that $\Theta^k G_k$ is even. We define the exponential $\exp(\Theta^1 G_1 + \dots + \Theta^q G_q)$ by the series expansion, which becomes just a finite polynomial, since $(\Theta^1 G_1 + \dots + \Theta^q G_q)^{q+1} = 0$. By convention we restrict the exponential also to $\mathscr{D}_g \otimes \mathscr{C}$. As an example we note that for $G \in L_{\mathscr{C}}$, ΘG even with $\Theta^2 = 0$,

$$\exp(t\Theta G) := I_g \otimes I|_{\mathscr{C}} + t\Theta G \tag{4.4}$$

for $t \in \mathbb{K}$, which we take to be either $\mathbb{R}$ or $\mathbb{C}$.

In order to investigate groups of exponentials, we need the following commutation relations.

Lemma 4.1. Let G and G' be homogeneous elements of $L_{\mathscr{C}}$, let $\Theta^2 = \Theta'^2 = 0$, and ΘG and $\Theta' G'$ be even. Then we find the relations

$$[\exp(t\Theta G), \exp(t'\Theta' G')]_- = -tt'\Theta\Theta'[G', G] = tt'\Theta'\Theta[G, G'] \tag{4.5}$$

and therefore, for $t, t', s, s' \in \mathbb{K}$,

$$\begin{aligned} &\exp(t\Theta G)\exp(t'\Theta' G')\exp(s\Theta G)\exp(s'\Theta' G') \\ &\quad = \exp[(t+s)\Theta G]\exp[(t'+s')\Theta' G')]\exp(st'\Theta\Theta'[G', G]). \end{aligned} \tag{4.6}$$

One-parameter groups $\{\exp t\Theta G | t \in \mathbb{K}\}$ of exponentials (4.4), with unit element $I_g \otimes I|_{\mathscr{C}}$, give representations of the additive group $\mathbb{K}$ and can be used to construct automorphism groups of the C(A)CR.

Next we investigate the multiple supercommutators of operators $\in L_{\mathscr{C}}$, which arise if one takes products of exponentials of the form (4.4). For each ordered multi-index $\{i\} = \{i_1, \ldots, i_p\}$, $1 \leq i_1 < \ldots < i_p \leq g$, we choose some homogeneous element $G_{\{i\}} \in L_{\mathscr{C}}$ of degree 0 (or 1) for even (or odd) p, and construct the commutative one-parameter group $\{T_{\{i\}}(t) | t \in \mathbb{K}\}$ of exponentials

$$T_{\{i\}}(t) := \exp(t\Theta_{\{i\}} G_{\{i\}}) \tag{4.7}$$

with $\Theta_{\{i\}} := \Theta_{i_1} \ldots \Theta_{i_p}$, which we call supertransformations. The $G_{\{i\}}$ will be called the generators of these transformations. If especially $G_{\{i\}} \in L^*_{\mathscr{C}}$ and

$$G_{\{i\}} \subseteq (-1)^{q+1} G^{\dagger}_{\{i\}} \qquad p = 2q \text{ or } 2q-1;\ q \in \mathbb{N} \tag{4.8}$$

where p denotes the 'length' of the multi-index $\{i\}$, then the supertransformations $T_{\{i\}}(t) \in \mathscr{D}_g \otimes A^*_{\mathscr{C}}$ become superunitary:

$$T^*_{\{i\}}(t) = T_{\{i\}}(-t) \qquad T^*_{\{i\}}(t) T_{\{i\}}(t) = T_{\{i\}}(t) T^*_{\{i\}}(t) = I_g \otimes I|_{\mathscr{C}} \tag{4.9}$$

for $t \in \mathbb{R}$. The generators $G_{\{i\}}$ need not be (anti)self-adjoint in general.

The set of finite products of supertransformations (4.7) forms a group $\mathscr{T}$ under the product (3.3), if we restrict it to $\mathscr{D}_g \otimes \mathscr{C}$. From (4.8) it follows that a supertransformation $T \in \mathscr{T}$ is superunitary, if its generators are (anti)symmetric.

Lemma 4.1 allows us to describe the group $\mathscr{T}$ with the help of finitely many parameters. For the case $g = 2$, for example, an easy iteration of (4.6) yields lemma 4.2.

Lemma 4.2. For $g = 2$ we choose G_1, G_2 and $G_{12} \in L_{\mathscr{C}}$, with G_{12} even, and G_1, G_2 odd. Every supertransformation $T \in \mathscr{T}$ can then be described by four real (or complex) parameters

$$\begin{aligned} T = {} & \exp(t_1\Theta_1 G_1) \exp(t_2\Theta_2 G_2) \exp(t_{12}\Theta_1\Theta_2[G_1, G_2]_+) \\ & \times \exp(s_{12}\Theta_1\Theta_2 G_{12}) t_1, t_2, t_{12}, s_{12} \in \mathbb{K}. \end{aligned} \tag{4.10}$$

T is superunitary iff the three generators are symmetric.

The number τ of independent real (or complex) parameters, which one must fix, in order to label the elements of the group $\mathscr{T}$, is finite due to the following lemma.

Lemma 4.3. τ is equal to the number of linearly independent multiple supercommutators of generators $G_{\{i_k\}}$ with disjoint multi-indices $\{i_k\}$, $k = 1, \ldots, q$.

In order to include the generators themselves we denote them as zero supercommutators. Note that the group $\mathscr{T}$ is constructed from a fixed family of generators $G_{\{i\}}$, $\{i\} = \{i_1, \ldots, i_p\}$, $1 \leq i_1 < \ldots < i_p \leq g$.

If one introduces more than one, but a finite number of generators for an ordered multi-index $\{i\}$, the group of finite products of supertransformations of type (4.7) is given by linear combinations of the generators belonging to the same multi-index, and by all linearly independent multiple supercommutators of such linear combinations.

5. Implementing automorphisms of C(A)CR

The superunitary transformations introduced above can be used to construct automorphisms of the C(A)CR. In order to simplify the notation, we introduce the operators

$$A^i_{k,0} := \begin{cases} B_k|_{\mathscr{C}_f} & \text{if } i=0, \\ C_k|_{\mathscr{C}_f} & \text{if } i=1,\, k=1,\dots,f \end{cases} \tag{5.1}$$

on an appropriate invariant domain $\mathscr{C}_f := \mathscr{C}_0 \otimes \mathscr{G}_f$, which is dense in $\mathscr{L}_f$. One may, for example, take $\mathscr{C}_0 = S(\mathbb{R}^f)$, the rapidly decreasing C^∞-functions. The following lemma is easily derived.

Lemma 5.1. Let $T_{\{i\}}(t)$ be the superunitary transformations defined in (4.6) to (4.8), and take $\mathscr{H} := \mathscr{L}_f$ and $\mathscr{C} := \mathscr{C}_f$. The C(A)CR are then conserved under these supertransformations in the sense that

$$[A^{i\prime}_k(t), A^{j\prime}_l(t)] = 0 \qquad [A^{i\prime}_k(t), A^{j\prime *}_l(t)] = \delta_{kl}\delta_{ij} I^f_g \qquad I^f_g := I_g \hat{\lambda} I|_{\mathscr{C}_f}$$

$$A^{i\prime}_k(t) := T_{\{i\}}(t) A^i_{k,0} T^*_{\{i\}}(t) = T_{\{i\}}(t) A^i_{k,0} T_{\{i\}}(-t) \tag{5.2}$$

$$t \in \mathbb{R} \qquad i,j = 0,1 \qquad k,l = 1,\dots,f.$$

It is simple to decompose such supertransformations following the notation of § 4.

Lemma 5.2. Let $A \in \mathrm{L}_{\mathscr{C}}$ and G' be a homogeneous element $\in L_{\mathscr{C}}$, $\Theta, \Theta' \in \mathscr{D}_g$ with $\Theta'^2 = 0$ and let $\Theta' G'$ be even. Then one obtains

$$\exp(\Theta' G')\Theta A \exp(-\Theta' G') = \Theta A + \Theta\Theta'[G', A]|_{D_g \otimes \mathscr{C}}. \tag{5.3}$$

By taking products of superunitary transformations (4.7)-(4.9) we define the following automorphism of the C(A)CR: Take $A \in \mathrm{L}_{\mathscr{C}}$ and let

$$T := \prod \exp(t_{\{i\}}\Theta_{\{i\}} G_{\{i\}}) = \prod T_{\{i\}}(t_{\{i\}}) \qquad t_{\{i\}} \in \mathbb{R} \tag{5.4}$$

with (anti)symmetric generators $G_{\{i\}} \in \mathrm{L}_{\mathscr{C}}$ like in (4.8). The product in (5.4) is ordered such that $T_{\{i_1 \dots i_p\}}$ stands to the left of $T_{\{j_1 \dots j_q\}}$ if $p > q$, or if $p = q$ and $i_1 > j_1$, or if $p = q$ and $i_1 = j_1$ and $i_2 > j_2$, etc. T is superunitary in the sense of (4.3). Using (5.4) we get the following expansion of TAT^* into multiple supercommutators

$$TAT^* = A + \sum t_{\{i_1\}} \dots t_{\{i_q\}} \Theta_{\{i_1\}} \dots \Theta_{\{i_q\}} [G_{\{i_q\}}, [\dots [G_{\{i_1\}}, A] \dots]] \tag{5.5}$$

where the sum runs over all possible choices of pairwise disjoint ordered multi-indices $\{i_k\}$, $k = 1, \dots, q$, which are then ordered according to the product (5.4): $\Theta_{\{i_k\}}$ is put to the left of $\Theta_{\{i_l\}}$ for $k < l$, if $T_{\{i_k\}}$ is put to the right of $T_{\{i_l\}}$ in the product (5.4).

Lemma 5.1 can now be iterated to lemma 5.3.

Lemma 5.3. We take $\mathscr{H} := \mathscr{L}_f$ and $\mathscr{C} := \mathscr{C}_f$, and obtain from the superunitary transformation T defined in (5.4) an automorphism of the fermionic oscillator representation of the C(A)CR by

$$\begin{aligned} &A^{i\prime}_k := T A^i_{k,0} T^* && i,j = 0,1;\ k,l = 1,\dots,f \\ &[A^{i\prime}_k, A^{j\prime}_l] = 0 && [A^{i\prime}_k, A^{j\prime *}_l] = \delta_{kl}\delta_{ij} I^f_g. \end{aligned} \tag{5.6}$$

Conversely, every automorphism of the C(A)CR can be implemented by a superunitary transformation in the following sense. In order to simplify the proofs, we treat first the case of one anticommuting parameter $\Theta = \Theta^*$.

Theorem 5.1. Assume that the linear subspace $\mathscr{C}_f := \mathscr{C}_0 \otimes \mathscr{G}_f$ of $\mathscr{L}_f$, where $\mathscr{C}_0$ is dense in $L^2(\mathbb{R}^f)$, is an invariant domain for the linear operators $A^i_{k,j}$ and their adjoints, i, $j=0,1$, $k=1,\ldots,f$; $j=0$ denotes the fermionic oscillator representation (5.1). $A^0_{k,1}$ is odd and $A^1_{k,1}$ is even. Define

$$\begin{aligned} A^i_k &:= A^i_{k,0} + \Theta A^i_{k,1} \qquad \Theta = \Theta^*;\ \Theta^2 = 0 \\ A^{i*}_k &= A^i_{k,0}{}^{\dagger} + (-1)^{i+1}\Theta A^i_{k,1}{}^{\dagger}|_{\mathscr{C}f} \qquad i=0,1;\ k=1,\ldots,f \end{aligned} \tag{5.7}$$

and assume that these coded operators fulfil the C(A)CR:

$$[A^i_k, A^j_l] = 0 \qquad [A^i_k, A^{j*}_l] = \delta_{kl}\delta_{ij} I^f_g. \tag{5.8}$$

It follows that there exists exactly one symmetric odd operator G on dom $G = \mathscr{C}_f$, such that $G\mathscr{C}_f \subseteq \mathscr{C}_f$, which generates the automorphism (5.7) in the sense

$$\begin{aligned} A^i_k &= \exp(\Theta G) A^i_{k,0} \exp(-\Theta G) \\ A^{i*}_k &= \exp(\Theta G) A^i_{k,0}{}^{\dagger} \exp(-\Theta G) \end{aligned} \tag{5.9}$$

which means that

$$A^i_{k,1} = [G, A^i_{k,0}] \qquad k=1,\ldots,f;\ i=0,1. \tag{5.10}$$

Proof. The 2^{2f} monomials

$$\begin{aligned} &C_{\{0;0\}} := I_f \qquad C_{\{p;0\}} := C^{\dagger}_{p_1}\ldots C^{\dagger}_{p_r} \qquad C_{\{0;q\}} := C_{q_1}\ldots C_{q_s} \\ &C_{\{p;q\}} := C^{\dagger}_{p_1}\ldots C^{\dagger}_{p_r} C_{q_1}\ldots C_{q_s} \qquad 1 \leqslant p_1 < \ldots < p_r \leqslant f;\ 1 \leqslant q_1 < \ldots < q_s \leqslant f \end{aligned} \tag{5.11}$$

form an operator basis for $\mathscr{G}_f$. Any element can be expanded as

$$\begin{aligned} &A^i_{k,1} = \sum_{\{p;q\}} A^{i,\{p,q\}}_{k,1} C_{\{p;q\}} \qquad i=0,1;\ k=1,\ldots,f \\ &\text{dom } A^{i,\{p;q\}\dagger}_{k,1} \supseteq \text{dom } A^{i,\{p;q\}}_{k,1} = \mathscr{C}_0 \supseteq A^{i,\{p;q\}}_{k,1}\mathscr{C}_0 \cup A^{i,\{p;q\}\dagger}_{k,1}\mathscr{C}_0. \end{aligned} \tag{5.12}$$

If one inserts (5.7) into (5.8), one obtains the supercommutation relations

$$[\Theta A^i_{k,1}, A^{j(*)}_{l,0}] + [A^i_{k,0}, (\Theta A^j_{l,1})^{(*)}] = 0 \qquad i,j=0,1;\ k,l=1,\ldots,f \tag{5.13}$$

where $(*)$ means that the involution is either done in both expressions or in none of them.

Using the (anti)commutation relations

$$[C_k, C_{\{p;q\}}] = (-1)^{l-1} C_{\{p_1\ldots p_{l-1}p_{l+1}\ldots p_r;q\}} \qquad \text{for } k = p_l \tag{5.14}$$

leads to

$$(-1)^l A^{1,\{p_1\ldots p_{l-1}p_{l+1}\ldots p_r;q\}}_{k,1} = (-1)^n A^{1,\{p_1\ldots p_{n-1}p_{n+1}\ldots p_r;q\}}_{m,1} \qquad k=p_l;\ m=p_n. \tag{5.15}$$

It follows that the generator

$$\begin{aligned} G &:= \sum_{\{p\}} [(-1)^{l-1} A^{1,\{p_1\ldots p_{l-1}p_{l+1}\ldots p_r;0\}}_{k,1} C_{\{p;0\}} + \text{adjoint}] \\ &\quad + \sum_{\{p;q\};s\geqslant 1} (-1)^{l-1} A^{1,\{p_1\ldots p_{l-1}p_{l+1}\ldots p_r;q_1\ldots q_s\}}_{k,1} C_{\{p;q\}} \qquad p_l = k \end{aligned} \tag{5.16}$$

is well defined. Using (5.13) one finds that G is symmetric and fulfills (5.10).

This result can be generalised to finitely many anticommuting parameters in the following way.

Theorem 5.2. Assume that the linear operators $A^i_{k,\{i\}} \in \mathrm{L}^*_{\mathscr{C}_f}$, $i=0,1$, $k=1,\ldots,f$, $\{i\} \equiv \{i_1,\ldots i_p\}$, $1 \leqslant i_1 < \ldots < i_p \leqslant g$, are defined on the invariant domain $\mathscr{C}_f := \mathscr{C}_0 \otimes \mathscr{G}_f$, where $\mathscr{C}_0$ is dense in $L^2(\mathbb{R}^f)$. $A^i_{k,0}$ denotes the fermionic oscillator representation (5.1). Let $\Theta_{\{i\}} A^i_{k,\{i\}}$ be even for $i=0$ and odd for $i=1$, where $\Theta_{\{i\}} := \Theta_{i_1} \ldots \Theta_{i_p}$. Assume that the coded operator

$$\begin{aligned} A^i_k &:= A^i_{k,0} + \sum_{\{i\}} \Theta_{\{i\}} A^i_{k,\{i\}} \qquad k=1,\ldots,f \\ A^{i*}_k &= A^{i\dagger}_{k,0} + \sum_{\{i\}} (-1)^{pi+q} \Theta_{\{i\}} A^{i\ \dagger}_{k,\{i\}}|_{\mathscr{C}_f} \\ \{i\} &= \{i_1,\ldots,i_{2q-1}\} \qquad \text{or} \qquad \{i\} = \{i_1,\ldots,i_{2q}\} \qquad q \in \mathbb{N} \end{aligned} \tag{5.17}$$

fulfill the C(A)CR,

$$[A^i_k, A^j_l] = 0 \qquad [A^i_k, A^{j*}_l] = \delta_{kl}\delta_{ij} I^f_g \qquad i,j=0,1;\ k,l=1,\ldots,f. \tag{5.18}$$

Then there exist (anti)symmetric operators $G_{\{i\}} \in \mathrm{L}^*_{\mathscr{C}_f}$,

$$\begin{aligned} &G_{\{i\}} \subseteq (-1)^{q+1} G^\dagger_{\{i\}} \\ &\{i\} = \{i_1,\ldots,i_{2q-1}\} \qquad \text{or} \qquad \{i\} = \{i_1,\ldots i_{2q}\} \qquad q \in \mathbb{N} \end{aligned} \tag{5.19}$$

which are odd or even, respectively, such that the superunitary transformation

$$T := \prod \exp(\Theta_{\{i\}} G_{\{i\}}) \qquad T^* T = TT^* = I^f_g \tag{5.20}$$

with the ordering defined in (5.4), implements the representation (5.17) of the C(A)CR in the sense that

$$A^i_k = TA^i_{k,0}T^* \qquad A^{i*}_k = TA^{i\ \dagger}_{k,0}T^* \qquad i=0,1;\ k=1,\ldots,f. \tag{5.21}$$

These generators are essentially unique in the following sense: For $\{i_1,\ldots,i_{2q-1}\}$, $G_{\{i\}}$ is unique; for $\{i_1,\ldots,i_{2q}\}$, the difference of two generators fulfilling (5.19)-(5.21) is proportional to a constant, $G_{\{i\}} - G'_{\{i\}} = i^{p/2+1} cI|_{\mathscr{C}_f}$, $c \in \mathbb{R}$, $\{i\} = \{i_1,\ldots,i_p\}$, if we assume that this difference is essentially self-adjoint on $\mathscr{C}_f := S(\mathbb{R}^f) \otimes \mathscr{G}_f$, where $S(\mathbb{R}^f)$ denotes the Schwarz space of rapidly decreasing C^∞ functions.

Corollary 5.1. Under the conditions of theorem 5.2, the generators fulfil the multiple supercommutation relations

$$A^i_k = A^i_{k,0} + \sum \Theta_{\{i_1\}} \ldots \Theta_{\{i_q\}} [G_{\{i_q\}}, [\ldots [G_{\{i_1\}}, A^i_{k,0}] \ldots]] \qquad i=0,1;\ k=1,\ldots,f \tag{5.22}$$

with the same ordering as in (5.5).

Remark. The C(A)CR (5.18) especially imply that the $A^i_{k,0}$ fulfil this algebra on $\mathscr{C}_f$. But the C(A)CR on $\mathscr{C}_f$ do not imply the unitary equivalence to the fermionic oscillator representation (5.1). Therefore we insert this special representation into the assumptions of theorems 5.1 and 5.2. Of course, it can be replaced by any unitarily equivalent representation.

The operator family, (5.17) and (5.18), will be called a coded representation of the C(A)CR on the invariant domain $\mathscr{C}_f$.

Proof of theorem 5.2. Inserting (5.17) into (5.18) one obtains, in a first step,

$$[\Theta_m A^i_{k,m}, A^{j(*)}_{l,0}] + [A^i_{k,0}, (\Theta_m A^j_{l,m})^{(*)}] = 0 \qquad i, j = 0, 1;\ k, l = 1, \ldots, f;\ m = 1, \ldots, g \tag{5.23}$$

similarly as in equations (5.13) in the previous proof. It therefore follows, according to (5.16), that there exist symmetric operators $G_m \in \mathrm{L}^*_{\mathscr{C}_f}$ such that

$$A^i_{k,m} = [G_m, A^i_{k,0}] \qquad i = 0, 1;\ k = 1, \ldots, f;\ m = 1, \ldots, g. \tag{5.24}$$

In the second step, we redefine

$$A^i_{k,\{m,n\}}{}' := A^i_{k,\{m,n\}} - [G_n, [G_m, A^i_{k,0}]] \tag{5.25}$$

and obtain the supercommutation relations

$$[\Theta_m \Theta_n A^i_{k,\{m,n\}}{}', A^{j(*)}_{l,0}] + [A^i_{k,0}, (\Theta_m \Theta_n A^j_{l,\{m,n\}}{}')^{(*)}] = 0 \qquad i, j = 0, 1;\ k, l = 1, \ldots, f. \tag{5.26}$$

Similarly as in the proof of theorem 5.1, one shows next the existence of symmetric operators $G_{\{m,n\}} \in \mathrm{L}^*_{\mathscr{C}_f}$, such that

$$A^i_{k,\{m,n\}}{}' = [G_{\{m,n\}}, A^i_{k,0}] \qquad 1 \leqslant m < n \leqslant g. \tag{5.27}$$

We continue this procedure. Before doing the $(p+1)$th step one has established the existence of (anti)symmetric generators $G_{\{j\}}$ for $\{j\} = \{j_1, \ldots, j_r\}$ with $r = 1, \ldots, p$, as stated in (5.19), and does again a redefinition

$$A^i_{k,\{i\}}{}' := A^i_{k,\{i\}} - \sum_{q \geqslant 2} [G_{\{i_q\}}, [\ldots [G_{\{i_1\}}, A^i_{k,0}] \ldots]] \qquad \{i\} = \{i_1, \ldots, i_{p+1}\} \tag{5.28}$$

where again the ordering of (5.5) is applied. The sum in (5.28) runs over all pairwise disjoint multi-indices $\{i_m\}$, $m = 1, \ldots, q$, $\{i_m\} = \{i_1, \ldots, i_r\}$, $r = 1, \ldots, p$, such that $\bigcup_{m=1}^{q} \{i_m\} = \{i\}$. Inserting (5.28) into (5.17) and (5.18), we obtain the supercommutation relations

$$[\Theta_{\{i\}} A^i_{k,\{i\}}{}', A^{j(*)}_{l,0}] + [A^i_{k,0}, (\Theta_{\{i\}} A^j_{l,\{i\}}{}')^{(*)}] = 0 \qquad i, j = 0, 1;\ k, l = 1, \ldots, f \tag{5.29}$$

from which one concludes again the existence of (anti)symmetric generators $G_{\{i\}}$, as proposed in (5.19). These generators fulfil the supercommutation relations

$$A^i_{k,\{i\}}{}' = [G_{\{i\}}, A^i_{k,0}] \qquad \{i\} = \{i_1, \ldots, i_{p+1}\}. \tag{5.30}$$

Finally, one obtains the expansion (5.22), which is equivalent to the result (5.21).

That the odd generators $G_{\{i\}}$ are unique follows from their (anti)symmetry. Essential uniqueness of the even generators is implied by the Kato condition [12, p 287], if we use that $(B_k + B^\dagger_k + iI_x)\mathscr{C}_0 = \mathscr{C}_0 = (B_k - B^\dagger_k + I_x)\mathscr{C}_0$ for $\mathscr{C}_0 := S(\mathbb{R}^f)$. Here I_x denotes the identity map of $L^2(\mathbb{R})$.

Example 5.1. For the case of two anticommuting parameters, $g = 2$,

$$\begin{aligned} &A^i_k := A^i_{k,0} + \Theta_1 A^i_{k,1} + \Theta_2 A^i_{k,2} + \Theta_1\Theta_2 A^i_{k,12} \\ &A^i_{k,m} = [G_m, A^i_{k,0}] \qquad m = 1, 2 \\ &A^i_{k,12} = [G_2, [G_1, A^i_{k,0}]] + [G_{12}, A^i_{k,0}] \qquad i = 0, 1;\ k = 1, \ldots, f \\ &T := \exp(\Theta_1\Theta_2 G_{12}) \exp(\Theta_2 G_2) \exp(\Theta_1 G_1) \end{aligned} \tag{5.31}$$

where G_1 and G_2 are odd and symmetric, and G_{12} is even and symmetric.

Example 5.2. In the case $g=3$, one obtains

$$A^i_{k,123}=[G_3,[G_2,[G_1,A^i_{k,0}]]]+[G_{23},[G_1,A^i_{k,0}]]-[G_{13},[G_2,A^i_{k,0}]]$$
$$+[G_{12},[G_3,A^i_{k,0}]]+[G_{123},A^i_{k,0}] \qquad i=0,1;\ k=1,\ldots,f \tag{5.32}$$

where G_{123} is odd and antisymmetric.

Example 5.3. If we use self-adjoint supercharges Q_n, $n=1,\ldots,N$, which obey the axioms (3.1), as generators $G_n := t_nQ_n|_{\mathscr{C}_f}$, we obtain the direct product of N groups of superunitary transformations of the form

$$\begin{aligned} T(t_1,\ldots,t_N) &:= \prod_{n=1}^{N}\exp(\Theta_nG_n)=\exp\left(\sum_{n=1}^{N}\Theta_nG_n\right)\\ T(t)T(t') &= T(t+t') \qquad t,t'\in\mathbb{R}^n. \end{aligned} \tag{5.33}$$

Example 5.4. If we insert an even number of self-adjoint supercharges Q_n, $n=1,\ldots,N$; $N=2M$; $M\in\mathbb{N}$, into the generators in the form

$$\begin{aligned} G_m &:= t_mQ_{M+m}+t_{M+m}Q_m|_{\mathscr{C}_f} \qquad && t_m,t_{M+m}\in\mathbb{R}\\ G_{M+m} &:= t_mQ_m-t_{M+m}Q_{M+m}|_{\mathscr{C}_f} && m=1,\ldots,M \end{aligned} \tag{5.34}$$

we obtain the following group of superunitary transformations:

$$\begin{aligned} T(s_1,\ldots,s_M;r_1,\ldots,r_M) &:= \exp\Bigg[\sum_{m=1}^{M}\Bigg(\mathrm{i}s_m\frac{Q_m+\mathrm{i}Q_{M+m}}{\sqrt{2}}(\Theta_m+\mathrm{i}\Theta_{M+m})\\ &\quad+\mathrm{i}s^*_m(\Theta_m-\mathrm{i}\Theta_{M+m})\frac{Q_m-\mathrm{i}Q_{M+m}}{\sqrt{2}}+2r_m\Theta_m\Theta_{M+m}H|_{\mathscr{C}_f}\Bigg)\Bigg] \end{aligned} \tag{5.35}$$

$$s_m := (t_m+\mathrm{i}t_{M+m})/\sqrt{2} \qquad r_m\in\mathbb{R}$$

$$T(s;0)=\prod_{n=1}^{N}\exp(\Theta_nG_n)=\exp\left(\sum_{n=1}^{N}\Theta_nG_n\right).$$

The group composition law

$$\begin{aligned} T(s;r)T(s';r') &= T(s+s';r_1+r'_1+2\,\mathrm{Im}\,s'_1s^*_1,\ldots,r_M+r'_M+2\,\mathrm{Im}\,s'_Ms^*_M)\\ &\quad s,s'\in\mathbb{C}^M;\ r,r'\in\mathbb{R}^M \end{aligned} \tag{5.36}$$

shows that we get the direct product of M groups of superunitary transformations, where each group is described by three real parameters.

6. Groups of automorphisms of the C(A)CR

For simplicity reasons, let us denote the families of coded operators $\{A^i_k\}$ and $\{A^i_{k,0}\}$ defined in (5.1), (5.17) and (5.18) by A and A_0, respectively. Theorem 5.2 implies that the coded C(A)CR representation A on $\mathscr{C}_f$ is implemented by an essentially unique superunitary transformation $T\in\mathscr{D}_g\otimes A^*_{\mathscr{C}_f}$, such that $A=TA_0T^*$ according to (5.20) and (5.21). Vice versa, any superunitary transformation T on $\mathscr{C}_f$ implements the coded C(A)CR representation $A=TA_0T^*$ on $\mathscr{C}_f$.

Let $A_j := T_j A_0 T_j^*$, $j = 1, 2$, be two coded representations of the C(A)CR, which are implemented by superunitary transformations T_j on $\mathscr{C}_f$. This implies that $A_2 = TA_1T^*$, where the superunitary transformation $T := T_2T_1^*$ on $\mathscr{C}_f$.

An automorphism τ of the C(A)CR on $\mathscr{C}_f$ is defined as a linear bijection of the associative superalgebra $\mathscr{D}_g \otimes A^*_{\mathscr{C}_f}$, which is compatible with the product of coded operators on $\mathscr{D}_g \otimes \mathscr{C}_f$ as well as with the involution (4.2) on $\mathscr{D}_g \otimes A^*_{\mathscr{C}_f}$. Therefore, a family of coded operators A fulfilling (5.17) and (5.18) is mapped onto A', which again satisfies the propositions of theorem 5.2. Due to this theorem, such an automorphism τ determines an equivalence class of superunitary transformations T, such that $\tau(A_0) = TA_0T^*$. Iff $\tau(A) = TAT^*$ holds for all C(A)CR representations A on $\mathscr{C}_f$, τ is called an inner automorphism of the C(A)CR on $\mathscr{C}_f$.

The inner automorphisms of C(A)CR on $\mathscr{C}_f$ form a group, which will be implemented by the group $\mathrm{U}_{\mathscr{C}_f}$ of superunitary transformations T on $\mathscr{C}_f$ due to the group homomorphism

$$T \to \tau(S) := TST^* \qquad S \in \mathscr{D}_g \otimes A^*_{\mathscr{C}_f} \qquad T \in \mathrm{U}_{\mathscr{C}_f}. \tag{6.1}$$

Theorem 5.2 tells us that the set of all coded representations of the C(A)CR on $\mathscr{C}_f$ is given by the family $\{TA_0T^* | T \in \mathrm{U}_{\mathscr{C}_f}\}$. The question, under which additional conditions an automorphism of the C(A)CR on $\mathscr{C}_f$ is an inner one, remains open.

These superunitary transformations can be combined with unitary transformations of the fermionic oscillator representation A_0, by transforming just the invariant domains appropriately.

Lemma 6.1. Let U' and U'' be unitary operators on $\mathscr{L}_f := L^2(\mathbb{R}^f) \otimes \mathscr{G}_f$, and denote by $\mathscr{C}'_f := U'\mathscr{C}_f$ and $C''_f := U''U'\mathscr{C}_f$ the transformed invariant domain $\mathscr{C}_f := \mathscr{C}_0 \otimes \mathscr{G}_f$ of the fermionic oscillator representation (5.1). With the superunitary transformations T' and T'' on $\mathscr{C}'_f$ and $\mathscr{C}''_f$, define

$$A'' := T''U''A'U''^{\dagger}T''^* \qquad A' := T'U'A_0U'^{\dagger}T'^*. \tag{6.2}$$

Here we use U' and U'' as a shorthand writing of the skew-symmetric tensor product $I_g \otimes U'$ and $I_g \otimes U''$. Then there is an appropriate superunitary transformation T on $\mathscr{C}''_f$ such that

$$A'' = TU''U'A_0U'^{\dagger}U''^{\dagger}T^*. \tag{6.3}$$

T is essentially unique in the sense of theorem 5.2. One may choose $T := T''U''T'U''^{\dagger}$.

Proof. A'' fulfills the propositions of theorem 5.2, with A_0 replaced by $U''U'A_0U'^{\dagger}U''^{\dagger}$.

The unitary transformations, which are used in lemma 6.1, need not be graded, but the superunitary transformations conserve grading in the sense of (4.3).

Example 6.1. If we start from example 5.4 with a Hamilton operator $H := Q_n^2$, $n = 1, \ldots, N$, and an invariant domain $\mathscr{C}_f \subseteq \mathrm{dom}\, H^{1/2}$, the group of superunitary transformations (5.35) implements an automorphism group of the C(A)CR with N anticommuting parameters. Following lemma 6.1 one can combine these superunitary transformations with the time evolution and form the product $\exp(\mathrm{i}tH)T(s, r)$, $s, r \in \mathbb{R}^{N/2}$, $t \in \mathbb{R}$.

Since the transformation $UAU\dagger$ of some coded C(A)CR representation A on $\mathscr{C}_f$ by any unitary operator U on $\mathscr{L}_f$ yields another coded C(A)CR representation on $U\mathscr{C}_f$, the inverse $(TU)^* = U^\dagger T^*$ will transform any coded C(A)CR representation A' on $\mathscr{C}'_f := U\mathscr{C}_f$ into another one on $\mathscr{C}_f$. T above is assumed to be superunitary on $\mathscr{C}'_f$, such that $U^\dagger TU$ is superunitary on $\mathscr{C}_f$ according to the following diagram:

$$\begin{tikzcd} A_0 \arrow[r, "U"] \arrow[d, "U^\dagger TU"'] & A'_0 \arrow[d, "T"] \\ A \arrow[r, "U"] & A' \end{tikzcd}$$

in which

$$A'_0 := UA_0U^\dagger \qquad A' := TA'_0T^* = UAU^\dagger \qquad A := U^\dagger A'U. \tag{6.4}$$

More generally, an automorphism τ of the C(A)CR on $\mathscr{C}'_f = U\mathscr{C}_f$ is combined with the unitary transformation U of $\mathscr{L}_f$ to the mapping $A' = \tau \circ \nu(A)$, which transforms any coded C(A)CR representation A on $\mathscr{C}_f$ to another one A' on $\mathscr{C}'_f$, with an implementing superunitary transformation T on $\mathscr{C}'_f$ such that $A' = TUAU^\dagger T^*$, where $\nu(A) := UAU^\dagger$. In general, T will depend on the choice of A.

The group homomorphism (6.1) can be generalised to products UTV with unitary operators U, V on $\mathscr{L}_f$ such that $U\mathscr{C}_f = V\mathscr{C}_f = \mathscr{C}_f$.

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